

Exact Multiparticle Amplitudes at Threshold in Large- N Component ϕ^4 Theory

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Abstract

I derive the set of recurrence relations between the amplitudes of multiparticle production at threshold in the standard large- N limit of the $O(N)$ -symmetric ϕ^4 theory which sums all relevant diagrams with arbitrary number of loops. I find an exact solution to the recurrence relations using the Gelfand–Dikiĭ representation of the diagonal resolvent of the Schrödinger operator. The result coincides with the tree amplitudes while the effect of loops is the renormalization of the coupling constant and mass. The form of the solution is due to the fact that the exact amplitude of the process $2 \rightarrow n$ vanishes at $n > 2$ on mass shell when averaged over the $O(N)$ -indices of incoming particles for dynamical reasons because of the cancellation between diagrams. I discuss some possible applications of large- N amplitudes, in particular, for the renormalon problem.

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1 Introduction

The problem of calculating amplitudes of multiparticle production at threshold has recently received a considerable interest [1]–[15]. The explicit results for the tree level amplitudes in ϕ^4 theory demonstrate the factorial growth which is expected due to the large amount of identical bosons in the final state. Further investigations of the tree amplitudes [2, 3] introduced a nice technique to deal with the problem. A very interesting property of the tree amplitudes which has been recently pointed out by Voloshin [4] is the nullification of the on-mass-shell amplitude of the process $2 \rightarrow n$ for $n > 4$. This nullification has been extended [5]–[14] to more general models and made it possible to calculate the amplitude $1 \rightarrow n$ at the one-loop level. A dynamical symmetry which may be responsible for the nullification has been discussed by Libanov et al. [15].

In the present paper I shall make an attempt to understand which properties of the tree and one-loop amplitudes could survive in the full theory which includes all loop diagrams. For this purpose I consider the N -component ϕ^4 theory which is exactly solvable in the large- N limit [16] when only the bubble diagrams contribute if there is no multiparticle production.

I derive the set of recurrence relations between the amplitudes of multiparticle production at threshold in the large- N limit of the $O(N)$ -symmetric ϕ^4 theory which sums all relevant diagrams with an arbitrary number of loops. I reduce these recurrence relations to a quantum mechanical problem which turns out to be possible due to the factorization at large N . I find an exact solution to the problem using the Gelfand–Dikiĭ representation of the diagonal resolvent of the Schrödinger operator. The result is quite similar to the tree amplitudes while the effect of loops is the renormalization of the coupling constant and mass. The form of the solution is due to the fact that the exact amplitude of the process $2 \rightarrow n$ vanishes for $n > 2$ at large N on mass shell when averaged over the $O(N)$ -indices of incoming particles. This nullification occurs for dynamical reasons because of the cancellation between diagrams.

This paper is organized as follows. Sect. 2 is devoted to the definitions and the description of the kinematics. In Sect. 3 I derive the set of the recurrence relations and transform it to a quantum mechanical problem. In Sect. 4 I present an exact solution and demonstrate that it satisfies the set of equations. The exact solution is possible since the diagonal resolvent of the Schrödinger operator has a very simple form. I extend the method of calculating the diagonal resolvent to the case of a more general potential of the Pöschl–Teller type. In Sect. 5 I consider some implications of the results and discuss why the calculation of large- N amplitudes for multiparticle production at threshold can be interesting for the renormalon problem. Appendix A contains a proof of the uniqueness of the solution.

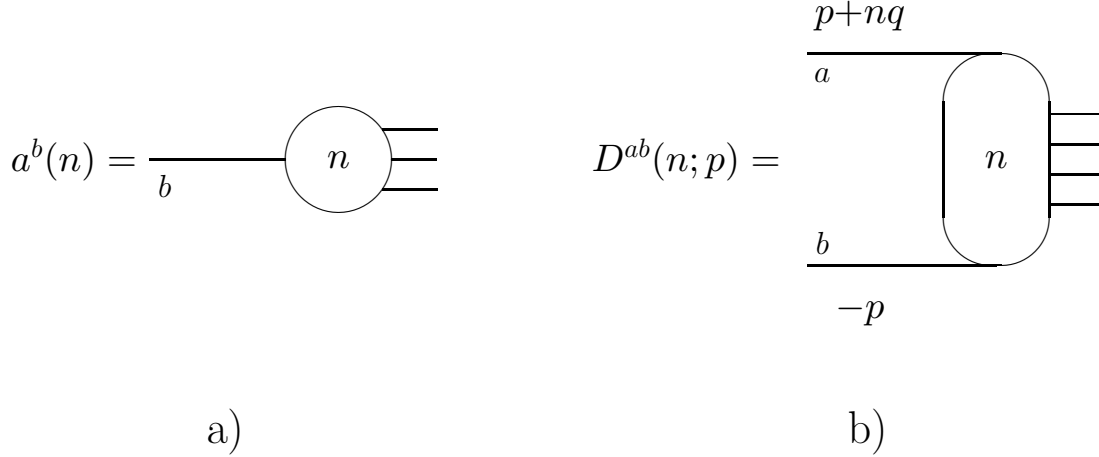


Figure 1: The graphic representations of the multiparticle amplitudes $a^b(n)$ (a)) and $D^{ab}(n; p)$ (b)).

2 The definitions

The $O(N)$ -symmetric ϕ^4 theory is defined in the 4-dimensional Minkowski space by the following Lagrangian

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi^b)(\partial_\mu \phi^b) - \frac{1}{2}m^2(\phi^b \phi^b) - \frac{1}{4}\lambda(\phi^b \phi^b)^2 \quad (2.1)$$

where $b = 1, \dots, N$ and the summation over repeated indices is implied.

We denote by $a_{b_1 \dots b_n}^b(n)$ the amplitude of production of n on-mass-shell particles with the $O(N)$ -indices $b_1 \dots b_n$ at rest by a (virtual) particle with the $O(N)$ -index b and the energy nm :

$$a_{b_1 \dots b_n}^b(n) = \langle b_1 \dots b_n | \phi^b(0) | 0 \rangle. \quad (2.2)$$

It is convenient to multiply this amplitude by the vectors ξ^{b_i} which results in a symmetrization over the $O(N)$ -indices of the produced particles and define

$$a^b(n) = a_{b_1 \dots b_n}^b(n) \xi^{b_1} \dots \xi^{b_n}. \quad (2.3)$$

The amplitude $a^b(n)$ is depicted graphically in Fig. 1a.

The second quantity which appears in the set of the recurrence relations between the amplitudes is the amplitude $D_{b_1 \dots b_n}^{ab}(n; p)$ of the process when two particles a and b with the 4-momenta $p + nq$ and $-p$, respectively, produce n on-mass-shell particles with the $O(N)$ -indices $b_1 \dots b_n$ and the equal 4-momenta $q = (m, 0)$ in the rest frame. We define again

$$D^{ab}(n; p) = D_{b_1 \dots b_n}^{ab}(n; p) \xi^{b_1} \dots \xi^{b_n}. \quad (2.4)$$

The amplitude $D^{ab}(n; p)$ is depicted graphically in Fig. 1b. The definitions (2.3) and (2.4) coincide for $N = 1$ with those originally introduced by Voloshin [1, 4].

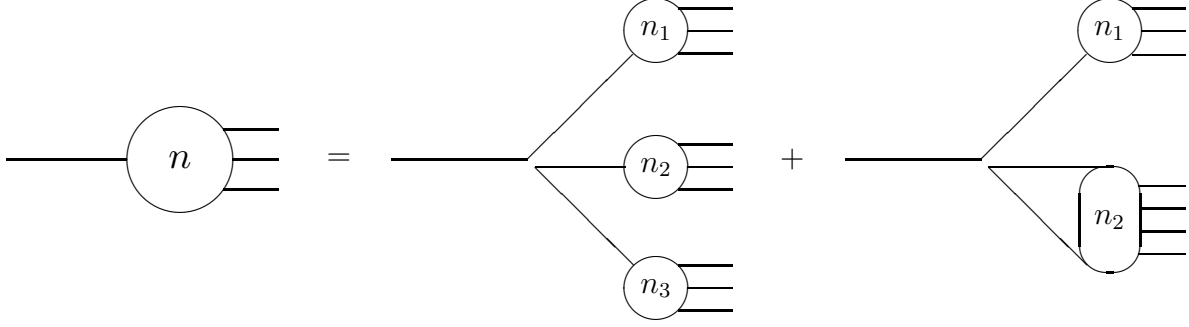


Figure 2: The recurrence relation for the amplitude $a^b(n)$. The analytical formula is given by Eq. (3.1).

It is easy to estimate at the tree level that

$$a^b(n) \sim \left(\lambda \xi^2\right)^{\frac{n-1}{2}} \xi^b \quad (2.5)$$

where ξ^2 stands for $\xi^a \xi^a$. Since

$$\lambda \sim \frac{1}{N} \quad (2.6)$$

in the large- N limit [16], we choose

$$\xi^2 \sim N \quad (2.7)$$

for all the amplitudes (2.5) to be of the same order in $1/N$.

3 Recurrence relations at large N

The condition (2.6) leaves at large- N only the diagrams with the largest possible number of sums over internal $O(N)$ -indices which propagate along closed loops of diagrams. The proper recurrence relations which extend the usual Schwinger–Dyson equations to the case when n particles are produced are depicted for $a^b(n)$ in Fig. 2. The continuous lines are associated with propagation of the $O(N)$ -indices. Each vertex is the sum of three possible permutations of the $O(N)$ -indices. This is taken into account in Fig. 2 by a combinatorial factor. The graphic notations becomes clear if one introduces the auxiliary field $\sigma(x) = \phi^2(x)$ which propagates in the empty space inside vertices.

The recurrence relation of Fig. 2 reads analytically as

$$\begin{aligned} a^a(n) = & \lambda m^{-6} \sum_{n_1, n_2, n_3 = \text{odd}} \delta_{n, n_1 + n_2 + n_3} \frac{n!}{n_1! n_2! n_3!} \frac{a^a(n_1) a^b(n_2) a^b(n_3)}{(n_1^2 - 1)(n_2^2 - 1)(n_3^2 - 1)} \\ & + \lambda m^{-2} \sum_{\substack{n_1 = \text{odd}, \\ n_2 = \text{even}}} \delta_{n, n_1 + n_2} \frac{n!}{n_1! n_2!} \frac{a^a(n_1)}{(n_1^2 - 1)} \int \frac{d^4 k}{(2\pi)^4} D^{bb}(n_2; k). \end{aligned} \quad (3.1)$$

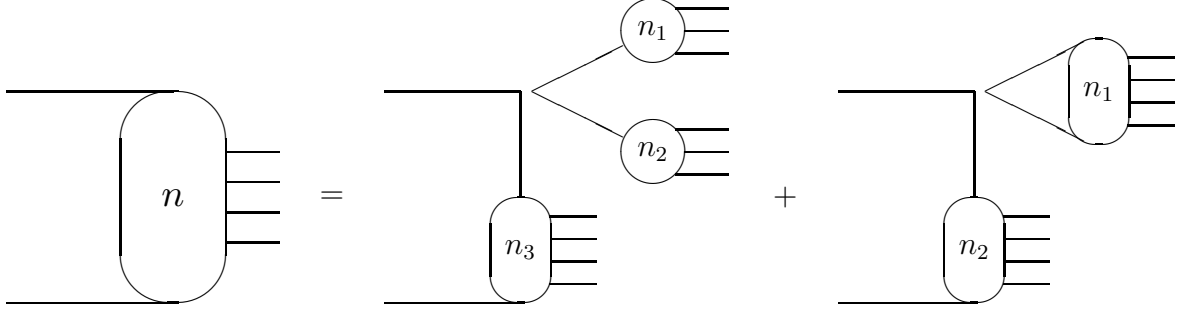


Figure 3: The recurrence relation for the amplitude $D^{aa}(n; p)$. The analytical formula is given by Eq. (3.2).

Notice that only the summed over the $O(N)$ -indices quantity $D^{bb}(n_2; k)$ enters this recurrence relation.

It is the property of large N that the recurrence relation for $D^{aa}(n; p)$ expresses it via a^b and D^{bb} again. The recurrence relation for $D^{aa}(n; p)$ is depicted in Fig. 3.

The recurrence relation of Fig. 3 reads analytically as²

$$D^{aa}(n) = \frac{\lambda m^{-4}}{(p + nq)^2 - m^2} \sum_{\substack{n_1, n_2 = \text{odd}, \\ n_3 = \text{even}}} \delta_{n, n_1 + n_2 + n_3} \frac{n!}{n_1! n_2! n_3!} \frac{a^b(n_1) a^b(n_2) D^{aa}(n_3; p)}{(n_1^2 - 1)(n_2^2 - 1)} \\ + \frac{\lambda}{(p + nq)^2 - m^2} \sum_{n_1, n_2 = \text{even}} \delta_{n, n_1 + n_2} \frac{n!}{n_1! n_2!} \int \frac{d^4 k}{(2\pi)^4} D^{bb}(n_1; k) D^{aa}(n_2; p). \quad (3.2)$$

Eqs. (3.1) and (3.2) look very similar to the ones [4, 12] for $N = 1$ at the one-loop level for $a(n)$ when one needs $D(n; p)$ only at the tree level so that the second term on the r.h.s. of Eq. (3.2) can be omitted.

To rewrite Eqs. (3.1) and (3.2) in a more convenient form, let us introduce, following the approach of Argyres et al. [2], the generating functions

$$\Phi^a(\tau) = m \xi^a e^{m\tau} + \sum_{n \geq 3} \frac{a^a(n)}{n!(n^2 - 1)} e^{nm\tau} m^{n-2} \quad (3.3)$$

and

$$D^{ab}(\tau; p) = \frac{\delta^{ab}}{p^2 - m^2} + \sum_{n = \text{even}} D^{ab}(n; p) \frac{1}{n!} e^{nm\tau} m^n. \quad (3.4)$$

Eqs. (3.1) and (3.2) can then be rewritten, respectively, as

$$\left\{ \frac{d^2}{d\tau^2} - m^2 - v(\tau) \right\} \Phi^a(\tau) = 0 \quad (3.5)$$

²Here and below the poles should be understood according to the Feynman prescription $m^2 \rightarrow m^2 - i0$.

and

$$\left\{ \frac{d^2}{d\tau^2} - \omega^2 - v(\tau) \right\} e^{\epsilon\tau} \frac{1}{N} D^{bb}(\tau; p) = e^{\epsilon\tau} \quad (3.6)$$

where ϵ is the energy component of p , $p = (\epsilon, \vec{p})$,

$$\omega = \sqrt{\vec{p}^2 + m^2} \quad (3.7)$$

and

$$v(\tau) = \lambda \Phi^2(\tau) + \lambda \int \frac{d^4 k}{(2\pi)^4} D^{bb}(\tau; k). \quad (3.8)$$

This transformation from Eqs. (3.1) and (3.2) to Eqs. (3.5) and (3.6) is quite similar to the one [4, 12] for $N = 1$ at the one-loop level. The extension to the $O(N)$ case was considered by Brown [3] at the tree level and by Smith [7] at the one loop level.

It is easy to understand using Eq. (3.4) that $v(\tau)$ which is defined by Eq. (3.8) is nothing but the sum of the matrix elements:

$$v(\tau) = \lambda \sum_{n=0}^{\infty} \langle b_1 \dots b_n | \phi^2(0) | 0 \rangle \xi^{b_1} \dots \xi^{b_n} \frac{m^n}{n!} e^{nm\tau}. \quad (3.9)$$

This formula as well as Eqs. (3.5), (3.6) can alternatively be derived by the use of the functional technique introduced for this problem by Brown [3] which relates τ to the time variable $t \equiv x_0$ of $\phi^a(x)$ by

$$\tau = it. \quad (3.10)$$

The fact that the matrix elements rather than operators themselves appear in Eqs. (3.5) and (3.6) is due to the factorization at large N (see e.g. Ref. [17]). It is the reason why Eqs. (3.5) and (3.6) form a closed set at large N .

The next step in the transformation of Eqs. (3.5), (3.6) and (3.8) is to pass for D^{ab} to the mixed representation — the coordinate in (imaginary) time and momentum for space. One defines

$$D_{\omega}^{ab}(\tau, \tau') = \int \frac{d\epsilon}{2\pi} e^{\epsilon(\tau - \tau')} D^{ab}(\tau; p). \quad (3.11)$$

In order to see that this quantity is indeed associated with the Fourier transform of amplitudes in energy, we notice that the insertion of Eq. (3.4) on the r.h.s. of Eq. (3.11) yields

$$D_{\omega}^{ab}(\tau, \tau') = \int \frac{d\epsilon}{2\pi} \sum_{n=0}^{\infty} e^{(\epsilon + nm)\tau - \epsilon\tau'} D^{ab}(n; p). \quad (3.12)$$

One recognizes now that $\epsilon + nm$ is the energy component of the 4-momentum $p + nq$ of the incoming particle with the $O(N)$ -index a while ϵ is that of p for b . In particular, the free propagator in the mixed representation is

$$D_{\omega}^{ab}(\tau, \tau') = \delta^{ab} \int \frac{d\epsilon}{2\pi} \frac{1}{(\epsilon^2 - \omega^2 - i0)} = \delta^{ab} \frac{1}{2\omega} e^{-\omega|\tau - \tau'|} \quad (\text{for } \lambda = 0). \quad (3.13)$$

The equations (3.6) and (3.8) can be finally rewritten in the mixed representation as follows:

$$\left\{ \frac{d^2}{d\tau^2} - \omega^2 - v(\tau) \right\} \frac{1}{N} D_\omega^{bb}(\tau, \tau') = -\delta(\tau - \tau') \quad (3.14)$$

and

$$v(\tau) = \lambda \Phi^2(\tau) + \frac{\lambda}{2\pi^2} \int_{m^2} d\omega \, \omega \sqrt{\omega^2 - m^2} D_\omega^{bb}(\tau, \tau). \quad (3.15)$$

To understand the meaning of the summed amplitude $D_\omega^{bb}(\tau, \tau')$, let us note that $D_\omega^{bc}(\tau, \tau')$ has in the index space the structure

$$D_\omega^{bc}(\tau, \tau') = \left(\delta^{bc} - \frac{\xi^b \xi^c}{\xi^2} \right) G_\omega^T(\tau, \tau') + \frac{\xi^b \xi^c}{\xi^2} G_\omega^S(\tau, \tau'). \quad (3.16)$$

The amplitudes G^T and G^S are associated, respectively, with the *tensor* and *singlet* $O(N)$ -states of two incoming particles. The averaged over the $O(N)$ -indices of two incoming particles amplitude is

$$\frac{1}{N} D_\omega^{bb}(\tau, \tau') = \left(1 - \frac{1}{N} \right) G_\omega^T(\tau, \tau') + \frac{1}{N} G_\omega^S(\tau, \tau') \quad (3.17)$$

while the generating function for the symmetrized over all the $n+2$ $O(N)$ -indices amplitudes is given by G^S :

$$D_\omega^{ab}(\tau, \tau') \frac{\xi^a \xi^b}{\xi^2} = G_\omega^S(\tau, \tau'). \quad (3.18)$$

One sees that the averaged amplitude which enter Eq. (3.15) coincides with G^T at large N .

Substituting (3.16) into Eq. (3.14), we rewrite it at large N as the following equation for $G_\omega^T(\tau, \tau')$:

$$\left\{ \frac{d^2}{d\tau^2} - \omega^2 - v(\tau) \right\} G_\omega^T(\tau, \tau') = -\delta(\tau - \tau'), \quad (3.19)$$

while $v(\tau)$ is related to $G_\omega^T(\tau, \tau)$ by

$$v(\tau) = \lambda \Phi^2(\tau) + \frac{\lambda N}{2\pi^2} \int_{m^2} d\omega \, \omega \sqrt{\omega^2 - m^2} G_\omega^T(\tau, \tau). \quad (3.20)$$

Therefore, only the tensor amplitude enters at large N .

Eqs. (3.19), (3.20) together with Eq. (3.5) form the closed set of equations which will be solved in the next section.

4 The exact solution

To solve the set of equations (3.5), (3.19) and (3.20), let us first look at Eq. (3.19) for given $v(\tau)$. This equation determines the Green function of the Schrödinger operator with

the potential $v(\tau)$ while τ plays the role of a 1-dimensional coordinate. In other language $G_\omega^T(\tau, \tau')$ is the matrix element of the resolvent

$$G_\omega^T(\tau, \tau') = \langle \tau | \frac{1}{-D^2 + \omega^2 + v} | \tau' \rangle \quad (4.1)$$

where D stands for $d/d\tau$ for brevity. One should take then the diagonal matrix element of the resolvent, $G_\omega^T(\tau, \tau)$, in order to substitute into Eq. (3.20) and to determine $v(\tau)$ versus Φ^2 .

The general solution of this problem for arbitrary v is given by the Gelfand–Dikiĭ formula [18]

$$G_\omega^T(\tau, \tau) = R_\omega[v] \equiv \sum_{l=0}^{\infty} \frac{R_l[v]}{\omega^{2l+1}} \quad (4.2)$$

where the differential polynomials $R_l[v]$ are determined recurrently by

$$R_l[v] = \frac{1}{2^l} \left(\frac{1}{2} D^2 - v - D^{-1} v D \right)^l \cdot \frac{1}{2} \quad (4.3)$$

and the inverse operator is

$$D^{-1}v(\tau) = \int_{-\infty}^{\tau} dx v(x). \quad (4.4)$$

Eq. (4.3) stems from the fact that $R_\omega[v]$ obeys the third order linear differential equation

$$\frac{1}{2} \left(\frac{1}{2} D^3 - Dv - vD \right) R_\omega[v] = \omega^2 D R_\omega[v]. \quad (4.5)$$

The polynomials $R_l[v]$ depend on v and its derivatives $v^{(s)} \equiv (D^s v)$. The first few polynomials are

$$\begin{aligned} R_0[v] &= \frac{1}{2}, \quad R_1[v] = -\frac{v}{4}, \quad R_2[v] = \frac{1}{16}(3v^2 - v''), \\ R_3[v] &= -\frac{1}{64}(10v^3 - 10vv'' - 5(v')^2 + v^{(4)}) \end{aligned} \quad (4.6)$$

while for τ -independent $v(\tau) = v_0$ one has

$$R_\omega[v_0] = \frac{1}{2\sqrt{\omega^2 + v_0}} \quad (v_0 = \text{const.}) \quad (4.7)$$

which agrees with Eq. (3.13) at $\tau = \tau'$.

Let us first show how our equations recover the known results [16] about the large- N limit of the $O(N)$ -symmetric ϕ^4 theory. We put $\xi^a \rightarrow 0$ in order to suppress the particle production. Then the solution for $v(\tau)$ is τ -independent. Using Eq. (4.7) and denoting

$$v_0 = m_R^2 - m^2, \quad (4.8)$$

one rewrites Eq. (3.20) in the form

$$m^2 = m_R^2 - \frac{\lambda N}{4\pi^2} \int_{m_R^2} d\omega \sqrt{\omega^2 - m_R^2}. \quad (4.9)$$

This formula exactly coincides with the standard expression for the bare mass, m , via the renormalized mass, m_R , at large N [16]. The meaning of this result is very simple: since $\xi^a = 0$, the only diagrams which are left in the recurrence relations of Fig. 3 are those with the bubble insertions to the propagator. They result solely in the mass renormalization.

It is convenient to perform the mass renormalization in Eqs. (3.5), (3.19) and (3.20) eliminating the τ -independent part of v . We introduce

$$v_R(\tau) = v(\tau) + m^2 - m_R^2 \quad (4.10)$$

after which everything is expressed in terms of m_R and $v_R(\tau)$:

$$\left\{ D^2 - m_R^2 - v_R(\tau) \right\} \Phi^a(\tau) = 0, \quad (4.11)$$

and

$$v_R(\tau) = \lambda \Phi^2(\tau) + \frac{\lambda N}{2\pi^2} \int_{m_R^2} d\omega \, \omega \sqrt{\omega^2 - m_R^2} \left(R_\omega[v_R] - \frac{1}{2\omega} \right), \quad (4.12)$$

where

$$\omega = \sqrt{\vec{p}^2 + m_R^2}. \quad (4.13)$$

We introduce also the renormalized coupling constant, λ_R , which is related to the bare one, λ , by

$$\frac{1}{\lambda} = \frac{1}{\lambda_R} - \frac{N}{8\pi^2} \int_{m_R^2} d\omega \frac{\sqrt{\omega^2 - m_R^2}}{\omega^2} \quad (4.14)$$

in order to rewrite Eq. (4.12) in the form

$$v_R(\tau) = \lambda_R \Phi^2(\tau) + \frac{\lambda_R N}{2\pi^2} \int_{m_R^2} d\omega \, \omega \sqrt{\omega^2 - m_R^2} \left(R_\omega[v_R] - \frac{R_1[v_R]}{\omega^3} - \frac{1}{2\omega} \right), \quad (4.15)$$

The integral over ω on the r.h.s. becomes convergent after the renormalizations.

Eq. (4.14) coincides with the standard renormalization of the coupling constant at large N [16]. The meaning of renormalization is that one chooses the bare quantities, m^2 and λ , to be dependent on the cut-off according to Eqs. (4.8) and (4.14) to make the renormalized ones, m_R^2 and λ_R , to be cut-off-independent.

I found the following *exact* solution to Eqs. (4.11) and (4.15):

$$G_\omega^T(\tau, \tau) = \frac{1}{2\omega} - \frac{\bar{\lambda}_R \Phi^2(\tau)}{4\omega(\omega^2 - m_R^2)}, \quad (4.16)$$

$$v_R(\tau) = \bar{\lambda}_R \Phi^2(\tau) \quad (4.17)$$

where

$$\bar{\lambda}_R = \frac{\lambda_R}{1 + \frac{\lambda_R N}{8\pi^2}} \quad (4.18)$$

differs from λ_R by a finite factor and $\Phi^a(\tau)$ reads

$$\Phi^a(\tau) = \frac{\xi^a m_R e^{m_R \tau}}{1 - \frac{\bar{\lambda}_R \xi^2}{8} e^{2m_R \tau}}. \quad (4.19)$$

Some comments concerning the exact solution are in order:

- i) Eqs. (4.16), (4.17) recover the large- N limit of the results [7] for the tree level $G_\omega^T(\tau, \tau)$ and the one-loop level $\Phi^a(\tau)$.
- ii) The fact that (4.16) has a pole only at $\omega^2 = m_R^2$ means the nullification of the tensor on-mass-shell amplitudes $2 \rightarrow n$ for $n > 2$. An analogous property holds [10] at $N = 1$ for the tree level amplitudes in the case of the sine-Gordon potential and for the ϕ^4 -theory with spontaneously broken symmetry [5].
- iii) The bare coupling λ is related to $\bar{\lambda}_R$ according to Eqs. (4.14) and (4.18) as follows

$$(\lambda N)^{-1} = (\bar{\lambda}_R N)^{-1} - \frac{1}{8\pi^2} \left(\int_{m_R^2} d\omega \omega^{-2} \sqrt{\omega^2 - m_R^2} + 1 \right) \quad (4.20)$$

The additional finite renormalization is familiar from the one-loop result [4] for $N = 1$.

- iii) Since $\bar{\lambda}_R$ becomes infinite at the negative value $\lambda_R = -8\pi^2/N$, $v_R(\tau)$ vanishes at this point when one goes around the pole singularity in (4.19). This is presumably associated with the double scaling limit of the 4-dimensional $O(N)$ -symmetric ϕ^4 theory [19].
- iiii) The large- N amplitude (4.19) is real which is related to the property of nullification discussed in the item ii). This is the difference with the $N = 1$ one-loop result [4].

Let us explicitly demonstrate the property listed in the item ii) which is crucial for our solution by the tree diagrams for the process $2 \rightarrow 4$. The diagrams which contribute to $D^{bb}(4; p)$ at large N are depicted in Fig. 4 together with the values of the 4-momenta (measured in the units of m) of each propagating virtual particle for the given kinematics when the incoming particles are on mass shell. Evaluating the contribution of each diagram of Fig. 4a, b and c and summing up, one gets

$$a) + b) + c) = -\frac{1}{4} + \frac{1}{8} + \frac{1}{8} = 0. \quad (4.21)$$

I performed also an analogous calculation for the process $2 \rightarrow 6$.

Let us now verify by direct calculations that (4.16), (4.17) and (4.14) is the solution to Eqs. (4.15), (4.11). The proof of the fact that this solution is unique for the problem of multiparticle production at threshold is presented in Appendix A.

It is instructive first to verify that Φ^2 is an eigenvector of the operator on the r.h.s. of Eq. (4.3):

$$\frac{1}{2} \left(\frac{1}{2} D^2 - v_R - D^{-1} v_R D \right) \Phi^2 = \frac{1}{2} D^{-1} \left(\frac{1}{2} D^3 - D v_R - v_R D \right) \Phi^2 = D^{-1} D m_R^2 \Phi^2 = m_R^2 \Phi^2. \quad (4.22)$$

This formula is derived using solely Eq. (4.11) for an arbitrary $v_R(\tau)$ and the fact that $\Phi^2(\tau)$ vanishes for $\tau \rightarrow \infty$.

We look now at Eq. (4.17) as an ansatz with some (yet to be determined) coefficient $\bar{\lambda}_R$. Then Eq. (4.22) transforms into the following second-order nonlinear differential equation for $v_R(\tau)$:

$$v_R'' - 3v_R^2 = 4m_R^2 v_R. \quad (4.23)$$

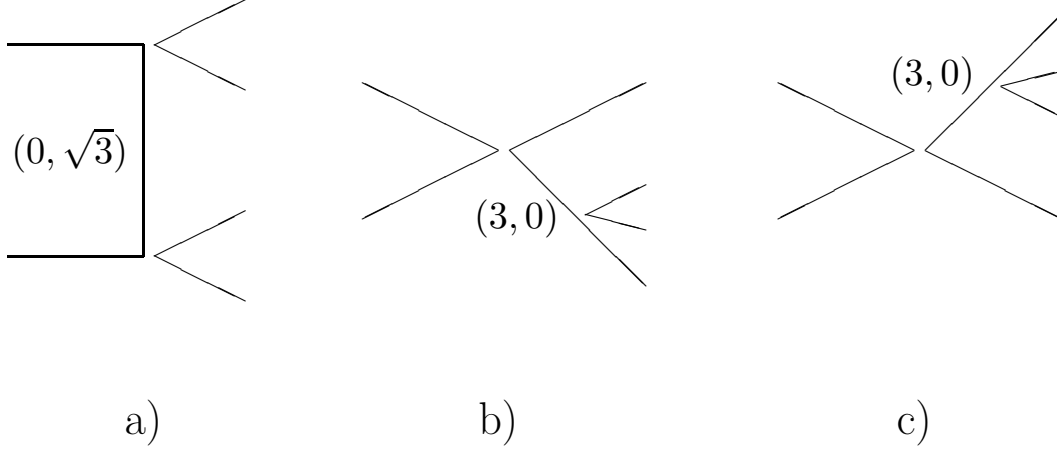


Figure 4: The tree diagrams for $D^{bb}(4; p)$ at large N . The components of the 4-momenta are given for the propagating virtual particles (in the units of mass) for the on-mass-shell kinematics. Only the energy and one space components are written since two other components vanish for all particles. The momenta of the incoming particles are $(2, \sqrt{3})$ and $(2, -\sqrt{3})$, respectively while that of the produced particles is $(1, 0)$. The cancellation of three diagrams is illustrated by Eq. (4.21).

Using Eq. (4.6) one can rewrite Eq. (4.23) as

$$R_2[v_R] = m_R^2 R_1[v_R]. \quad (4.24)$$

This equation is remarkable because we can apply again the same operator as in Eq. (4.22) to both sides of Eq. (4.24) to get, according to Eq. (4.3), $R_3[v_R]$ on the l.h.s and $R_2[v_R]$ on the r.h.s. which is proportional in our case to v_R . Applying the operator several times, we get

$$R_l[v_R] = -\frac{1}{4} m_R^{2l-2} v_R \quad (4.25)$$

which gives the formula (4.16) by using Eq. (4.2).

Substituting Eq. (4.16) into Eq. (4.15), one finds that it is satisfied for any τ providing λ_R and $\bar{\lambda}_R$ are related by Eq. (4.18).

It is still left to solve Eq. (4.23) to find an explicit solution for $v_R(\tau)$. In fact Eq. (4.23) is well known and has the solution which coincides (for $\alpha < 0$) with the profile of the one-soliton solution of the Korteweg-de Vries equation at a fixed value of time [20]:

$$v_R(\tau) = 2D^2 \log(1 - \alpha e^{2m_R\tau}) = \frac{8\alpha m_R^2 e^{2m_R\tau}}{(1 - \alpha e^{2m_R\tau})^2} = \frac{2m_R^2}{\sinh^2(m_R\tau + \frac{1}{2} \log \alpha)}, \quad (4.26)$$

where α is an arbitrary constant which is related to this fixed value of time and $4m_R^2$ plays the role of the asymptotic soliton speed. Since $v_R(\tau)$ is related to $\Phi^2(\tau)$ by Eq. (4.17),

one gets

$$\Phi^a(\tau) = \sqrt{\frac{8\alpha}{\bar{\lambda}_R \xi^2}} \frac{\xi^a m_R e^{m_R \tau}}{(1 - \alpha e^{2m_R \tau})}. \quad (4.27)$$

The expression (4.27) is familiar from the tree level results [2, 3]. It satisfies the classical equation for the $O(N)$ -symmetric ϕ^4 theory with the coupling constant $\bar{\lambda}_R$:

$$\{D^2 - m_R^2 - \bar{\lambda}_R \Phi^2(\tau)\} \Phi^a(\tau) = 0, \quad (4.28)$$

which can be easily obtained substituting (4.17) in Eq. (4.11).

The value of α can be fixed comparing with the $\xi \rightarrow 0$ limit when the multiparticle production is suppressed. In this limit only the diagrams with one insertion of the bubble chain contribute to the recurrence relation of Fig. 2. These diagrams lead, as is discussed above, to the mass renormalization which is given by Eq. (4.8). In the given case they change the coefficient m in front of the exponential in the first term on the r.h.s. of Eq. (3.3) to be m_R . Therefore, the expansion of Φ^a in ξ should start from

$$\Phi^a(\tau) = \xi^a m_R e^{m_R \tau} + \xi^a \mathcal{O}(\xi^2) \quad (4.29)$$

which fixes $\alpha = \bar{\lambda}_R \xi^2 / 8$ and results in Eq. (4.19).

Our method of finding the diagonal resolvent can be extended to the equation

$$\left\{ \frac{d^2}{d\tau^2} - \omega^2 - 3v_R(\tau) \right\} G_\omega(\tau, \tau') = -\delta(\tau - \tau'), \quad (4.30)$$

where the operator differs from the one in Eq. (3.19) by the extra coefficient 3 in front of $v_R(\tau)$. This operator emerges [4] in the $N = 1$ case. First we express $G_\omega(\tau, \tau)$ via the diagonal resolvent of the operator on the l.h.s. of Eq. (4.30):

$$G_\omega(\tau, \tau) = R_\omega[3v_R] \quad (4.31)$$

for v_R being the solution of Eq. (4.23). We then verify that

$$\begin{aligned} \frac{1}{2} \left(\frac{1}{2} D^2 - 3v_R - 3D^{-1}v_R D \right) v_R &= m_R^2 v_R - \frac{3}{2} v_R^2, \\ \frac{1}{2} \left(\frac{1}{2} D^2 - 3v_R - 3D^{-1}v_R D \right) v_R^2 &= 4m_R^2 v_R^2. \end{aligned} \quad (4.32)$$

The second equation is remarkable because it shows that v_R^2 is the eigenvector. Applying the operator several times, one proves by induction that

$$R_{l+1}[3v_R] = -\frac{3v_R}{4} m_R^{2l} + (4^l - 1) \frac{3v_R^2}{8} m_R^{2l-2} \quad (4.33)$$

which results using Eq. (4.31) in the formula

$$R_\omega[3v_R] = \frac{1}{2\omega} - \frac{3v_R(\tau)}{4\omega(\omega^2 - m_R^2)} + \frac{9v_R^2(\tau)}{8\omega(\omega^2 - m_R^2)(\omega^2 - 4m_R^2)}. \quad (4.34)$$

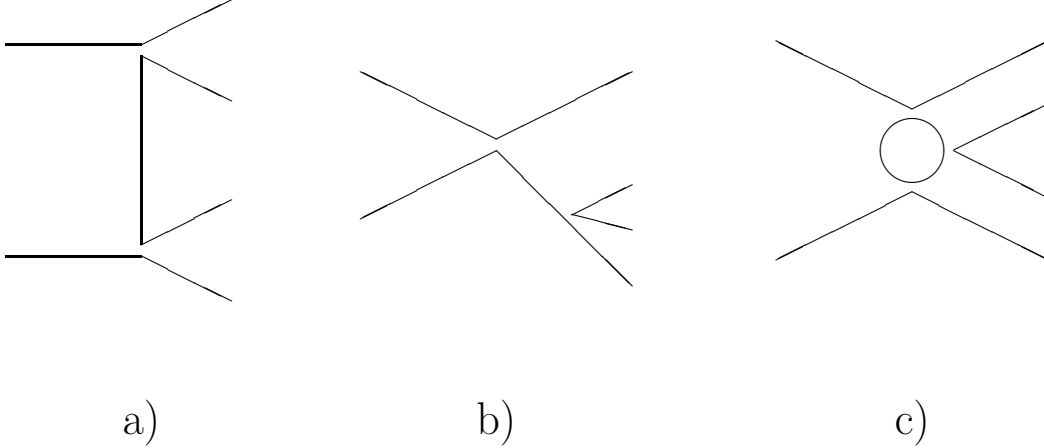


Figure 5: Some large- N diagrams for $G^S(4; p)$.

It is easy to extend the results of the previous paragraph to the resolvent $R_\omega[\frac{s(s+1)}{2}v_R]$ for $s > 2$ which also appears in applications [6, 9, 13]. One notices that

$$\frac{1}{2} \left(\frac{1}{2} D^2 - \frac{s(s+1)}{2} v_R - \frac{s(s+1)}{2} D^{-1} v_R D \right) v_R^l = l^2 m_R^2 v_R^l + \frac{l(l+1) - s(s+1)}{2(l+1)} \left(l + \frac{1}{2} \right) v_R^{l+1}. \quad (4.35)$$

Therefore, the highest power of v_R which emerges in $R_\omega[\frac{s(s+1)}{2}v_R]$ is v_R^s since the second term on the r.h.s. vanishes for $l = s$.

5 Discussion

An interesting property of the multiparticle production at threshold in the $O(N)$ -symmetric ϕ^4 theory at large N is that the amplitude $a^b(n)$ forms a closed set of exact equations together with the tensor amplitude $G^T(n; p)$. The $1 \rightarrow n$ amplitude $a^b(n)$ is real in a perfect agreement with unitarity since non-trivial intermediate states are suppressed by $1/N$ (like the wave-function renormalization starts from the order $1/N$). This is similar to the case of $N = 1$ with spontaneously broken symmetry where the $1 \rightarrow n$ amplitude is real to all orders [8]. As far as $G^T(n; p)$ on mass shell is concern, it vanishes for $n > 2$ for dynamical reasons and is real for the process $2 \rightarrow 2$ due to kinematics. I would say that the explicit results for $a^b(n)$ and $G^T(n; p)$ are not very instructive for these reasons for evaluations of the cross section. Analogously, the contribution of classical solutions to the partition function is exponentially suppressed by N at large N .

The singlet amplitude $G^S(n; p)$ reveals, on the contrary, a surprisingly non-trivial behavior even at large N . Some of the diagrams which contribute to $G^S(4; p)$ are depicted in Fig. 5. The combinatorics is now different from that of the diagrams of Fig. 4 so there is no cancellation of tree diagrams for $G^S(4; p)$ on mass shell which happens $n \geq 6$ according

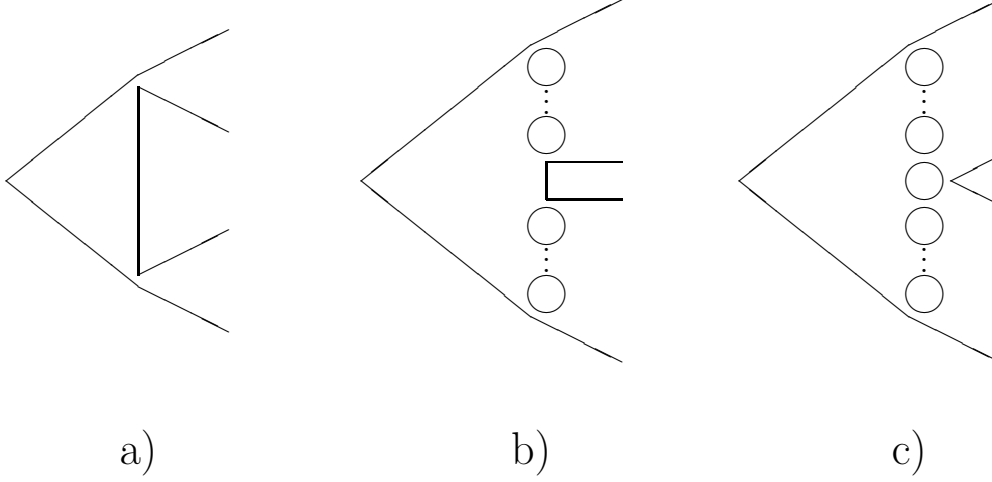


Figure 6: The diagrammatic representation of some diagrams for $G^S(4;p)$ at large N integrated over d^4p . The diagrams b) and c) are of the renormalon type while \vdots stands for the bubble chain.

to the explicit result [7] for $G^S(4;p)$ at the tree level. Each of vertices can be “dressed” with bubble chains and each of the lines of outgoing particles can be substituted by the exact amplitude $a^b(n)$ to obtain a diagram with more produced particles.

One can integrate $G^S(4;p)$ over the 4-momentum p to obtain diagrams of the vertex type. Some of them are depicted in Fig. 6. The most interesting are the diagrams of Fig. 6b, c which are of the type of renormalons [21] and should behave to k -th order of perturbation theory as $k!$. It would be very interesting for this reason to calculate $G^S(n;p)$ exactly at large N . This calculation should be simpler than at $N = 1$ since the factorization is no longer valid at finite N when one should deal with the whole chain of the Schwinger–Dyson equations for multipoint Green functions which were analyzed at the tree level by Smith [11].

Another interesting model to investigate is the case of the *matrix* Higgs field which is described by the Lagrangian

$$\mathcal{L} = \frac{1}{2} \text{tr}(\partial_\mu \phi)^2 - \frac{m^2}{2} \text{tr} \phi^2 - \frac{\lambda_3}{3} \text{tr} \phi^3 - \frac{\lambda_4}{4} \text{tr} \phi^4 \quad (5.1)$$

where $\phi^{ij}(x)$ is generically $N \times N$ matrix. The model greatly simplifies as $N \rightarrow \infty$ at fixed $\lambda_3^2 N$ or $\lambda_4 N$ when only the planar diagrams survive similar to the ‘t Hooft limit of QCD [22]. I have checked that the nullification of on-mass-shell amplitudes holds in the case of the matrix cubic interaction for $2 \rightarrow 3$ at large- N . This looks quite similar to what is observed in this paper for the $O(N)$ -case. The large- N limit of the matrix ϕ^4 theory while being simpler than the $N = 1$ case is, however, quite non-trivial like the large- N limit of QCD and the cross section will be no longer suppressed by $1/N$.

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Appendix A The proof of uniqueness of the solution

Let us apply to Eq. (4.15) the operator $\frac{1}{2}(D^2 - v_R - D^{-1}v_R D)$ which enters the recurrence relation (4.3). The crucial observation is that Φ^2 is an eigenvector of this operator:

$$\frac{1}{2}\left(\frac{1}{2}D^2 - v_R - D^{-1}v_R D\right)\Phi^2 = m_R^2\Phi^2. \quad (\text{A.1})$$

This formula is derived in Eq. (4.22) for an arbitrary $v_R(\tau)$ using only Eq. (4.11) and the asymptotics of $\Phi^2(\tau)$ for $\tau \rightarrow \infty$. Therefore, one obtains from Eq. (4.15)

$$-4R_2[v_R] = \lambda_R m_R^2 \Phi^2(\tau) + \frac{\lambda_R N}{2\pi^2} \int_{m_R^2} d\omega \omega^3 \sqrt{\omega^2 - m_R^2} \left(R_\omega[v_R] - \frac{R_2[v_R]}{\omega^5} - \frac{R_1[v_R]}{\omega^3} - \frac{1}{2\omega} \right). \quad (\text{A.2})$$

Here we used Eq. (4.3) and the fact that $R_\omega[v_R]$ satisfies the linear equation (4.5).

The application of the operator successively l times yields

$$-4R_{l+1}[v_R] = \lambda_R m_R^{2l} \Phi^2(\tau) + \frac{\lambda_R N}{2\pi^2} \int_{m_R^2} d\omega \omega^{2l} \sqrt{\omega^2 - m_R^2} \left(\omega R_\omega[v_R] - \sum_{s=0}^{l+1} \frac{R_s[v_R]}{\omega^{2s}} \right). \quad (\text{A.3})$$

Notice that the integral over ω is convergent for any l . Multiplying by ν^{2l+2} and summing over l we get finally

$$\begin{aligned} -4\nu R_\nu[v_R] + 2 &= \frac{\lambda_R \Phi^2(\tau)}{\nu^2 - m_R^2} \\ &+ \frac{\lambda_R N}{2\pi^2} \int_{m_R^2} d\omega \frac{\sqrt{\omega^2 - m_R^2}}{\omega^2} \left(\frac{\omega^3 R_\omega[v_R] - \nu^3 R_\nu[v_R]}{\nu^2 - \omega^2} + \frac{1}{2} \right). \end{aligned} \quad (\text{A.4})$$

The last expression can be conveniently rewritten as the contour integral

$$-4\nu R_\nu[v_R] + 2 = \frac{\lambda_R \Phi^2(\tau)}{\nu^2 - m_R^2} + \frac{\lambda_R N}{2\pi^2} \int_{m_R^2} d\omega \frac{\sqrt{\omega^2 - m_R^2}}{\omega^2} \oint_{C_1} \frac{dz}{2\pi i} \frac{z \sqrt{z} R_{\sqrt{z}}[v_R]}{(\nu^2 - z)(z - \omega^2)} \quad (\text{A.5})$$

where C_1 encircles the poles at $z = \omega^2$, ν^2 and infinity leaving outside singularities of $\sqrt{z} R_{\sqrt{z}}[v_R]$.

Now the idea is to solve Eq. (A.5) for $\nu R_\nu[v_R]$ as a function of ν rather than as a functional of $v_R(\tau)$. The analytic properties of $\sqrt{z} R_{\sqrt{z}}[v_R]$ as a function of z are known for our process of multiparticle production — it may have poles and branch cuts on the real axis since the operator in Eq. (3.19) is self-adjoint. They may start at $z = (n/2)^2 m_R^2$ which is associated with creation of n particles. In addition, $\sqrt{z} R_{\sqrt{z}}[v_R]$ is complex conjugate in the upper and lower half-planes since it is real in the interval $-m_R^2 < z < m_R^2$ (below any thresholds). One can always compress the contour C_1 in Eq. (A.5) to encircle the poles and branch cuts of $\sqrt{z} R_{\sqrt{z}}[v_R]$ which yields

$$-4\nu R_\nu[v_R] + 2 = \frac{\lambda_R \Phi^2(\tau)}{\nu^2 - m_R^2} + \frac{\lambda_R N}{2\pi^2} \int_{m_R^2} d\omega \frac{\sqrt{\omega^2 - m_R^2}}{\omega^2} \int \frac{dx}{\pi} \frac{x \operatorname{Im} \sqrt{x} R_{\sqrt{x}}[v_R]}{(\nu^2 - x)(x - \omega^2)} \quad (\text{A.6})$$

where the integral is along the support of $\text{Im } \sqrt{z} R_{\sqrt{z}}[v_R]$.

Let us now take the imaginary part of Eq. (A.6) for real ν . One gets

$$-4 \text{Im } \nu R_{\nu}[v_R] = \lambda_R \Phi^2(\tau) \pi \delta(\nu^2 - m_R^2) + \frac{\lambda_R N}{2\pi^2} \int_{m_R^2} d\omega \frac{\sqrt{\omega^2 - m_R^2}}{\omega^2} \frac{\nu^2 \text{Im } \nu R_{\nu}[v_R]}{(\nu^2 - \omega^2)}. \quad (\text{A.7})$$

This linear equation for $\text{Im } \nu R_{\nu}[v_R]$ has the unique solution

$$\text{Im } \nu R_{\nu}[v_R] = \bar{\lambda}_R \Phi^2 \pi \delta(\nu^2 - m_R^2) \quad (\text{A.8})$$

where $\bar{\lambda}_R$ is given by Eq. (4.18). Using the requirement that $\nu R_{\nu}[0] = 1/2$, we unambiguously obtain

$$R_{\nu}[v_R] = \frac{1}{2\nu} - \frac{\bar{\lambda}_R \Phi^2(\tau)}{4\nu(\nu^2 - m_R^2)} \quad (\text{A.9})$$

which coincides with (4.16).

This completes the proof of the uniqueness of the solution (4.16), (4.17), (4.18), (4.19).

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